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# Buckling and Postbuckling of Delaminated Composite Sandwich Beams

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**Consider a sandwich plate with balanced or unbalanced anisotropic composite laminated faces and an ideally orthotropic honeycomb core. The paper presents elastic buckling and postbuckling analysis of an axially loaded plate with an across-the-width delamination symmetrically located at the interface of the upper face and core. Since the plate undergoes cylindrical bending deformations on the postbuckling states for the cases considered, only one-dimensional formulations are employed. The transverse deflection and bending moment of the postbuckling solutions are obtained by applying the one-dimensional formulations. The explicit closed-form expressions of the critical buckling load and energy release rate are derived based on this postbuckling solution. Because there is no such general solution presented in the literature, verification is done by some special cases such as delaminated composites (without core), perfect sandwiches (without delamination), and thin-film delaminations. The effects of core, face, and delamination length on the buckling load, the delamination growth, and the ultimate axial load capacity of the delaminated composite sandwiches are also discussed in this paper.**

## Nomenclature

| | |
|---|---|
| $A_i$ | $= 1/(A_{11})_i, i = 1,2,3$ |
| $A_{ij}$ | = extensional stiffness of laminated composite |
| $a$ | = half-length of delamination |
| $B_i$ | $= (B_{11}/A_{11})_i, i = 1,2,3$ |
| $B_{ij}$ | = coupling stiffness of laminated composite |
| $c$ | = core thickness of sandwich |
| $D_i$ | $= (D_{11} - B_{11}^2/A_{11})_i, i = 1,2,3$ |
| $D_{ij}$ | = bending stiffness of laminated composite |
| $E_z$ | = Young's modulus in $z$ direction |
| $f$ | = face thickness when two faces are of equal thickness |
| $f_1, f_2$ | = thicknesses of upper and lower faces |
| $G$ | = energy release rate |
| $G_{xz}, G_{yz}$ | = transverse shear moduli in $x$-$z$ and $y$-$z$ planes |
| $G_0$ | = critical energy release rate |
| $l$ | = half-length of composite sandwich |
| $M_i$ | = bending moment of region $i$, $i = 1,2,3$ |
| $M_x, M_y, M_{xy}$ | = resultant moments |
| $N_x, N_y, N_{xy}$ | = resultant forces |
| $n$ | = number of layers including upper and lower faces, Eq. (6) |
| $P$ | = compressive axial force applied at the ends |
| $P_{cr}$ | = buckling load of composite sandwich |
| $P_{cr}^c$ | = buckling load of composite laminate |
| $P_i$ | = compressive axial forces of region $i, i = 1,2,3$ |
| $\bar{P}_i$ | = $P_i$ at critical buckling load |
| $\bar{Q}_{ij}$ | = transformed reduced in-plane stiffness of a single layer |
| $Q_x, Q_y$ | = resultant transverse shear forces |
| $S$ | = $cG_{xz}$, transverse shear stiffness |
| $U$ | = strain energy |
| $u, v, w$ | = displacements in $x$, $y$, and $z$ directions |
| $u_i, v_i, w_i$ | = displacements of region $i$ |
| $u_0, v_0, w_0$ | = displacements of the plane $z = 0$ |
| $W$ | = external work |
| $x, y, z$ | = Cartesian coordinates |
| $z_k$ | = $z$ value of the bottom of $k$th layer |
| $\Gamma$ | = amplitude of deflection |
| $\gamma_{xy}, \gamma_{xz}, \gamma_{yz}$ | = shear strains in $x$-$y$, $x$-$z$, and $y$-$z$ planes |
| $\epsilon_x, \epsilon_y, \epsilon_z$ | = normal strains in $x$, $y$, and $z$ directions |
| $\epsilon_{x_0}, \epsilon_{y_0}, \epsilon_{z_0}$ | = normal strains of the plane $z = 0$ |
| $\theta$ | = orientation of fiber measured counterclockwise from $x$ axis |
| $\kappa_x, \kappa_y, \kappa_{xy}$ | = curvatures |
| $\lambda_i$ | $= [P_i/D_i]^{1/2}, i = 2,3$ |
| $\bar{\lambda}_i$ | = $\lambda_i$ at critical buckling load |
| $\lambda_1$ | $= (P_1/\{D_1[1 - (P_1/S)]\})^{1/2}$ |
| $\Pi$ | = total potential energy |
| $\sigma_x, \sigma_y, \sigma_z$ | = normal stresses in $x$, $y$, and $z$ directions |
| $\sigma_{xy}, \sigma_{xz}, \sigma_{yz}$ | = shear stresses in $x$-$y$, $x$-$z$, and $y$-$z$ planes |

## I. Introduction

SANDWICH construction has been used in aeronautical applications for more than 40 years, since it has many advantages, for example, high bending stiffness, good weight savings, good surface finish, good fatigue properties, good thermal and acoustical insulation, etc.[1] Today, there is a renewed interest in using sandwich structures due to the introduction of new materials such as laminated composites for the faces of sandwich panels, which offer a long awaited capability with both high stiffness and low specific weight. Similarly, new materials for the core are now available, such as nonmetallic honeycombs and plastic foams. However, these new materials also induce some new problems that need to be solved. One of the most frequently encountered problems in composite laminates is interface cracking, sometimes known as delamination. For composite sandwiches, there is an extra interface between the face and core that may be weaker than those in layered composite faces. Delaminations may occur due to a variety of reasons such as low energy impact, manufacturing defects, or high stress concentrations at geometric or material discontinuities (e.g., the well-known free-edge effects). The presence of delaminations is of major concern, especially in compressively loaded components where delaminations may grow under fatigue loading by out-of-plane distortion.

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Buckling and postbuckling analysis for a delaminated composite has been studied by several researchers.[2,3] Although there are some papers dealing with the buckling problems of composite sandwiches,[4-8] to the authors' knowledge, none of them consider delamination. In the present analysis, similar to the approach of Chai et al.[2] and Yin et al.,[3] we develop a one-dimensional mathematical model for the delaminated composite sandwiches. General expressions of the postbuckling solution, as well as the algebraic equation governing the amplitude of deflection, are obtained in this paper. A characteristic equation for the critical buckling load is then derived by considering the stage immediately after the buckling of the delaminated sandwich. The energy release rate is obtained explicitly by differentiating the potential energy of the delaminated sandwich with respect to the length of delamination. The stability of delamination growth and the ultimate axial load capacity of the sandwiches are discussed based on this result. Some special cases like delaminated composites (without core), perfect sandwiches (without delamination), and thin-film delaminations can also be reduced from the present solution, which are verified by the solutions given in the literature.[1,3,9]

## II. Formulations of Sandwich Plates

Consider a sandwich plate with balanced or unbalanced anisotropic composite laminated faces and an ideally orthotropic honeycomb core. To simplify the analysis, the following assumptions are usually made for sandwiches.[10] 1) The faces are relatively thinner than the depth of the core. 2) The transverse shear deformation may be considered in the faces. However, the contribution of the transverse shear forces by the faces may be neglected as compared with those contributed by the core. Hence, we may say that the faces carry loads in their own planes only. 3) The core has direct stiffness normal to the faces and shear stiffness in planes normal to the faces, but no other stiffness. 4) The face-to-core bond insures that any displacement in the core adjacent to the faces is reproduced exactly in faces, and vice versa. According to these assumptions, the faces take almost all of the in-plane loadings and bending moments, whereas the core undergoes only transverse shear and normal forces. Thus, the stress-strain relations for the orthotropic core are

$$\sigma_x = \sigma_y = \tau_{xy} = 0, \quad \sigma_z = E_z \epsilon_z$$
$$\tau_{xz} = G_{xz}\gamma_{xz}, \quad \tau_{yz} = G_{yz}\gamma_{yz} \tag{1a}$$

where $E_z$, $G_{xz}$, and $G_{yz}$, are, respectively, the Young's modulus in the $z$ direction and the transverse shear moduli in $x$-$z$ and $y$-$z$ planes. If the honeycomb cellular core is considered, $G_{xz}$, $G_{yz}$, and $E_z$ should be interpreted into the effective moduli that are obtained by treating honeycomb as a homogeneous orthotropic continuum and can be related to the geometry and actual material properties of the core.[11] Lacking three of the stress components, the strain-displacement equation and the equilibrium equation can be expressed as

$$\epsilon_z = \frac{\partial w}{\partial z}, \quad \gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}, \quad \gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \tag{1b}$$

$$\frac{\partial \tau_{xz}}{\partial z} = 0, \quad \frac{\partial \tau_{yz}}{\partial z} = 0, \quad \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = 0 \tag{1c}$$

By Eq. (1c) it follows that $\tau_{xz}$ and $\tau_{yz}$ are functions of $x$ and $y$ only and that

$$\sigma_z = -z\left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y}\right) + \sigma_{z_0}$$

where $\sigma_{z_0}$ is the value of $\sigma_z$ at $z = 0$ and is thus a function of $x$ and $y$ only. By substituting the stresses for the strains in Eq. (1a) and integrating Eq. (1b), the following relations for the displacements are obtained:

$$u = \frac{z^3}{6E_z}\frac{\partial}{\partial x}\left(G_{xz}\frac{\partial \gamma_{xz}}{\partial x} + G_{yz}\frac{\partial \gamma_{yz}}{\partial y}\right) - \frac{z^2}{2E_z}\frac{\partial \sigma_{z_0}}{\partial x} + z\left(\gamma_{xz} - \frac{\partial w_0}{\partial x}\right) + u_0$$
$$v = \frac{z^3}{6E_z}\frac{\partial}{\partial y}\left(G_{xz}\frac{\partial \gamma_{xz}}{\partial x} + G_{yz}\frac{\partial \gamma_{yz}}{\partial y}\right) - \frac{z^2}{2E_z}\frac{\partial \sigma_{z_0}}{\partial y} + z\left(\gamma_{yz} - \frac{\partial w_0}{\partial y}\right) + v_0 \tag{2a}$$
$$w = \frac{-z^2}{2E_z}\frac{\partial}{\partial x}\left(G_{xz}\frac{\partial \gamma_{xz}}{\partial x} + G_{yz}\frac{\partial \gamma_{yz}}{\partial y}\right) + z\frac{\sigma_{z_0}}{E_z} + w_0$$

where $u_0$, $v_0$, and $w_0$ are the displacements of the plane $z = 0$, and $\sigma_{z_0}$, $\gamma_{xz}$, and $\gamma_{yz}$ are functions of $x$ and $y$ only. The transverse normal stiffness $E_z$ of the core is assumed to be infinitely large because the transverse normal strains are usually negligible for sandwiches having honeycomb cores.[1] Based on this assumption and Eq. (2a), the core displacements can be expressed as[4]

$$u = u_0 + z\left(\gamma_{xz} - \frac{\partial w_0}{\partial x}\right)$$
$$v = v_0 + z\left(\gamma_{yz} - \frac{\partial w_0}{\partial y}\right) \tag{2b}$$
$$w = w_0$$

which is the model usually given by the shear deformation plate theory.[12,13]

Since the delamination buckling and postbuckling are the main concern in this paper, a consideration of finite deformation should be included. If the deformations are considered as functions of $x$, $y$, and $z$ (the position of points in the unstrained configuration), the Lagrangian strains[14] $\epsilon_x$, $\epsilon_y$, and $\gamma_{xy}$ can be evaluated with the aid of Eq. (2b) as

$$\epsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 = \epsilon_{x_0} + z\kappa_x$$
$$\epsilon_y = \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 = \epsilon_{y_0} + z\kappa_y \tag{3}$$
$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} = \gamma_{xy_0} + z\kappa_{xy}$$

where

$$\epsilon_{x_0} = \frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2$$
$$\epsilon_{y_0} = \frac{\partial v_0}{\partial y} + \frac{1}{2}\left(\frac{\partial w_0}{\partial y}\right)^2 \tag{4a}$$
$$\gamma_{xy_0} = \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} + \frac{\partial w_0}{\partial x}\frac{\partial w_0}{\partial y}$$

$$\kappa_x = \frac{\partial}{\partial x}\left(\gamma_{xz} - \frac{\partial w_0}{\partial x}\right)$$
$$\kappa_y = \frac{\partial}{\partial y}\left(\gamma_{yz} - \frac{\partial w_0}{\partial y}\right) \tag{4b}$$
$$\kappa_{xy} = \frac{\partial}{\partial y}\left(\gamma_{xz} - \frac{\partial w_0}{\partial x}\right) + \frac{\partial}{\partial x}\left(\gamma_{yz} - \frac{\partial w_0}{\partial y}\right)$$

Since the faces are firmly bonded to the core, the displacements in the core adjacent to the faces are reproduced exactly in the faces. That is, the displacements of the faces adjacent to the core can be obtained by substituting $z = \pm c/2$ into Eq. (2b), where + and − are for lower and upper faces, respectively. Therefore, it is natural for us to assume that the displacements of the faces have the same form as those of the core described in Eq. (2b). By this face displacement representation, the requirement of displacement continuity across the interface of face and core[15] will then be satisfied automatically. It should be noted that the plane of $z = 0$ for the face displacements is still the midplane of the core, not the midplane of the face. Moreover, the transverse shear deformations are also included in this expression even though the transverse shear forces may be neglected as compared with those contributed by the core since the thicknesses of the faces are considered to be relatively thinner than the depth of the core.

With the same form as the core displacements given in Eq. (2b), the resultant forces and moments contributed by the faces can then be defined by the way similar to the classical lamination theory[16] as

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{12} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_{x_0} \\ \epsilon_{y_0} \\ \gamma_{xy_0} \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} \tag{5}$$

where

$$\begin{aligned} A_{ij} &= \sum_{k=1}^{n} (\bar{Q}_{ij})_k (z_k - z_{k-1}) \\ B_{ij} &= \frac{1}{2} \sum_{k=1}^{n} (\bar{Q}_{ij})_k (z_k^2 - z_{k-1}^2) \\ D_{ij} &= \frac{1}{3} \sum_{k=1}^{n} (\bar{Q}_{ij})_k (z_k^3 - z_{k-1}^3) \end{aligned} \tag{6}$$

Unlike the classical lamination theory in which $A_{ij}$, $B_{ij}$, and $D_{ij}$ are calculated based on the coordinate where $z = 0$ is the middle surface of the laminate, here the plane $z = 0$ is located on the midsurface of the core. Hence, the usual conclusions that $B_{ij} = 0$ for symmetric laminates and that $A_{16} = A_{26} = D_{16} = D_{26} = 0$ for antisymmetric laminates may not be valid in each face but may be valid when the overall sandwich is symmetric or antisymmetric.

The shear forces contributed by the core are

$$\begin{Bmatrix} Q_x \\ Q_y \end{Bmatrix} = c \begin{Bmatrix} G_{xz}\gamma_{xz} \\ G_{yz}\gamma_{yz} \end{Bmatrix} \tag{7}$$

As we have said previously, the transverse shear forces contributed by the faces are negligible as compared with those contributed by the core. The expression shown in Eq. (7) may represent the total transverse shear force of the sandwich plates.

The equilibrium equations for the buckled plate expressed by the resultant forces are[11]

$$\begin{aligned} &\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0 \\ &\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = 0 \\ &\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + N_x \frac{\partial^2 w}{\partial x^2} + 2N_{xy}\frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2} = 0 \\ &\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} = Q_x \\ &\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} = Q_y \end{aligned} \tag{8}$$

By using Eqs. (4–7), the five equilibrium equations for the buckled sandwich plate given in Eq. (8) can be written in terms of five unknowns, $u_0$, $v_0$, $w_0$, $\gamma_{xz}$, and $\gamma_{yz}$. For one-dimensional problems, the five equations can be further reduced to three equations that are

$$\frac{\partial N_x}{\partial x} = 0 \tag{9a}$$

$$N_x \frac{d^2 w}{dx^2} + cG_{xz}\frac{d\gamma_{xz}}{dx} = 0 \tag{9b}$$

$$B_{11}\frac{d}{dx}\left[\frac{du_0}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2\right] + D_{11}\frac{d^2}{dx^2}\left(\gamma_{xz} - \frac{dw}{dx}\right) = cG_{xz}\gamma_{xz} \tag{9c}$$

where

$$N_x = A_{11}\left[\frac{du_0}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2\right] + B_{11}\frac{d}{dx}\left(\gamma_{xz} - \frac{dw}{dx}\right) \tag{9d}$$

Equation (9a) reveals that $N_x$ is a constant throughout the plate and is equal to the compressive axial load $P$ applied at the ends, i.e.,

$$N_x = -P \tag{10}$$

From Eqs. (9b) and (9d), we have

$$\begin{aligned} \frac{d\gamma_{xz}}{dx} &= \frac{P}{S}\frac{d^2 w}{dx^2} \\ \frac{du_0}{dx} &= -\frac{P}{A_{11}} - \frac{1}{2}\left(\frac{dw}{dx}\right)^2 + \frac{B_{11}}{A_{11}}\left(1 - \frac{P}{S}\right)\frac{d^2 w}{dx^2} \end{aligned} \tag{11}$$

where $S = cG_{xz}$ represents transverse shear stiffness. Substituting Eq. (11) into Eq. (9c), we obtain the governing equation for buckled sandwich beam, which is expressed by only one parameter $w$,

$$\frac{d^2 w}{dx^2} + \lambda^2 w = c_1 x + c_2 \tag{12a}$$

where

$$\lambda^2 = \frac{P}{(D_{11} - B_{11}^2/A_{11})(1 - P/S)} \tag{12b}$$

and $c_1$ and $c_2$ are integration constants that will be determined by the boundary conditions.

## III. Delamination Buckling and Postbuckling

We now consider a sandwich plate containing an interface crack (delamination) lying between the upper face and core. The plate has a constant width between two lateral edges and is subjected to compressive axial loads $P$ at the clamped ends $x = \pm l$. As shown in Fig. 1, the crack extends over an interval $-a \le x \le a$ and runs across the whole width of the plate. The delaminated sandwich starts to buckle when the axial load reaches a critical value. If appropriate boundary conditions are maintained along the lateral edges, the plate undergoes cylindrical bending deformation in the postbuckling state. Hence, the one-dimensional formulations presented in Eqs. (9–12) can be employed. To analyze the delaminated sandwich, the entire plate is separated into three regions as shown in Fig. 1, and each region is considered as a sandwich beam

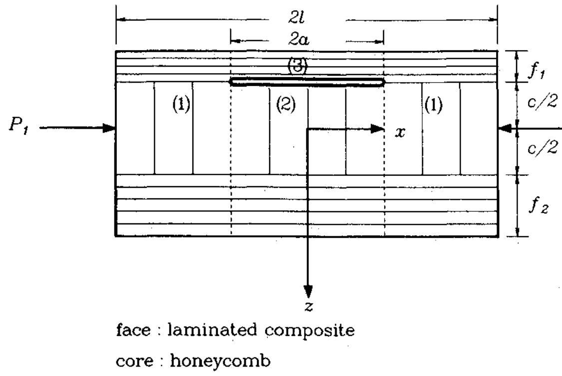


**Fig. 1 Delaminated composite sandwich.**

(region 1 and 2) or laminated beam (region 3). The governing equation, Eq. (12), is now employed for these three regions. The term containing $S$ vanishes for regions 2 and 3 since $\tau_{xz} = 0$ (or $\gamma_{xz} = 0$) along the crack surfaces provided that the crack remains completely open. By using the symmetry condition with respect to the midpoint, the clamped boundary conditions at the two ends, and the continuity of deflection and slope at the crack tip, one may obtain the following expressions for the transverse deflection in each region:

$$w_1 = \Gamma[1 - \cos\lambda_1(l - x)], \quad a \le x \le l$$

$$w_2 = \Gamma\left[\frac{\lambda_1 \sin\lambda_1(l-a)}{\lambda_2 \sin\lambda_2 a}(\cos\lambda_2 x - \cos\lambda_2 a) + 1 - \cos\lambda_1(l-a)\right], \quad 0 \le x \le a \tag{13a}$$

$$w_3 = \Gamma\left[\frac{\lambda_1 \sin\lambda_1(l-a)}{\lambda_3 \sin\lambda_3 a}(\cos\lambda_3 x - \cos\lambda_3 a) + 1 - \cos\lambda_1(l-a)\right], \quad 0 \le x \le a$$

where

$$\lambda_1^2 = \frac{P_1}{D_1(1 - P_1/S)}, \quad \lambda_2^2 = \frac{P_2}{D_2}, \quad \lambda_3^2 = \frac{P_3}{D_3} \tag{13b}$$

Note that in Eq. (13b) the simplified notations

$$A_i = \frac{1}{(A_{11})_i}, \quad B_i = \left(\frac{B_{11}}{A_{11}}\right)_i, \quad D_i = \left(D_{11} - \frac{B_{11}^2}{A_{11}}\right)_i$$
$$i = 1,2,3 \tag{13c}$$

have been used and the subscripts $i$, $i = 1,2,3$, denote the region number. The $P_i$ ($i = 1,2,3$) is the compressive force of region $i$, and $\Gamma$ represents the amplitude of deflection; $P_1 = P$ is given if the applied axial force is known. The three unknowns, $P_2$, $P_3$, and $\Gamma$, will be determined later on. The coordinate system used here is chosen with $x$ measured from the central line of delamination and $z = 0$ laid on the middle plane of the core.

To determine $P_2$, $P_3$, and $\Gamma$, we consider the equilibrium of forces and moments at the crack tip $x = a$ and the compatibility of regions 2 and 3, which are

$$P_1 = P_2 + P_3$$
$$M_1 = M_2 + M_3 \tag{14a}$$
$$u_2\left(a, -\frac{c}{2}\right) = u_3\left(a, -\frac{c}{2}\right)$$

where

$$M_i = -P_i\left(B_i + \frac{1}{\lambda_i^2}\frac{d^2 w_i}{dx^2}\right), \quad i = 1,2,3$$
$$u_i = -A_i P_i x - \frac{1}{2}\int_0^x \left(\frac{dw_i}{dx}\right)^2 dx + (B_i - z)\frac{dw_i}{dx}, \quad i = 2,3 \tag{14b}$$

In the previous text, the first equation of Eq. (14b) is derived by using the first equations of Eq. (4a) and Eq. (4b), the fourth equation of Eq. (5), and Eq. (11), whereas the second equation of Eq. (14b) comes from the first equation of Eq. (2b) and the second equation of Eq. (11). Substituting Eqs. (13a) and (14b) into (14a), we obtain

$$\gamma^2\left(\frac{1}{\sin^2\lambda_2 a} - \frac{1}{\sin^2\lambda_3 a} - \frac{1}{\lambda_2 a \tan\lambda_2 a} + \frac{1}{\lambda_3 a \tan\lambda_3 a}\right)$$
$$+ 2\gamma(B_2 - B_3)/a + A_2 P_1 - (A_2 + A_3)P_3 = 0 \tag{15a}$$
$$\gamma = \tfrac{1}{2}[(B_2 - B_1)P_1 + (B_3 - B_2)P_3] \times \left(\frac{P_1}{\lambda_1 \tan\lambda_1(l-a)} + \frac{P_1 - P_3}{\lambda_2 \tan\lambda_2 a} + \frac{P_3}{\lambda_3 \tan\lambda_3 a}\right)^{-1}$$

where

$$\gamma = \tfrac{1}{2}\Gamma\lambda_1 \sin\lambda_1(l - a) \tag{15b}$$

For any given applied axial force $P(= P_1)$ that exceeds the critical buckling load, the system of Eq. (15) furnishes two algebraic equations for the unknowns $P_3$ and $\gamma$ since $\lambda_1$, $\lambda_2$, and $\lambda_3$ are related to $P_1$ and $P_3$ by Eq. (13b). By Eqs. (15b), (13), and (14b), solution of this system of equations yields the unknown amplitude $\Gamma$, the transverse deflections $w_i$, and the bending moments $M_i$ of postbuckling solution. To determine the buckling load $P_{cr}$, we consider the stage immediately after the buckling of the delaminated sandwich. At this stage, the deflection amplitude $\Gamma$ is small and the second-order term $\gamma^2$ in the first equation of Eq. (15a) is negligible in comparison with the remaining terms. Hence,

$$P_3 \approx \frac{A_2}{A_2 + A_3}P_1 + \frac{2(B_2 - B_3)}{a(A_2 + A_3)}\gamma \tag{16}$$

Substituting Eq. (16) into the second equation of Eq. (15a), we obtain

$$\left[B_2 - B_1 + \frac{A_2(B_3 - B_2)}{A_2 + A_3}\right]P_1 - 2\gamma\left[\frac{P_1}{\lambda_1 \tan\lambda_1(l-a)} + \frac{\bar{P}_2}{\bar{\lambda}_2 \tan\bar{\lambda}_2 a} + \frac{\bar{P}_3}{\bar{\lambda}_3 \tan\bar{\lambda}_3 a} + \frac{(B_2 - B_3)^2}{a(A_2 + A_3)}\right] \approx 0 \tag{17a}$$

where

$$\bar{P}_2 = \frac{A_3}{A_2 + A_3}P_1, \quad \bar{P}_3 = \frac{A_2}{A_2 + A_3}P_1$$
$$\bar{\lambda}_2^2 = \frac{\bar{P}_2}{D_2}, \quad \bar{\lambda}_3^2 = \frac{\bar{P}_3}{D_3} \tag{17b}$$

The first term of Eq. (17a) is trivial since it can be proved to be identical to zero by using Eq. (13c), $(A_{11})_1(A_{11})_2 + (A_{11})_3$, and $(B_{11})_1 = (B_{11})_2 + (B_{11})_3$. When the compressive load $P(= P_1)$ reaches the critical buckling load, the amplitude of deflection $\Gamma$ (hence $\gamma$) could be arbitrary. Therefore, the coefficient of $\gamma$ in Eq. (17a) should be zero, which yields the following characteristic equation for $P_{cr}$:

$$P_{cr} = (D_2 + D_3 - D_1) \times \left[ \frac{a}{\bar{\lambda}_1 \tan \bar{\lambda}_1(l - a)} + \frac{ka}{\bar{\lambda}_2 \tan \bar{\lambda}_2 a} + \frac{(1-k)a}{\bar{\lambda}_3 \tan \bar{\lambda}_3 a} \right]^{-1} \tag{18a}$$

where

$$k = \frac{A_3}{A_2 + A_3}$$
$$\bar{\lambda}_1^2 = \frac{P_{cr}}{D_1(1 - P_{cr}/S)}, \quad \bar{\lambda}_2^2 = \frac{kP_{cr}}{D_2}, \quad \bar{\lambda}_3^2 = \frac{(1-k)P_{cr}}{D_3} \tag{18b}$$

Note that the equality $[-(B_2 - B_3)^2/(A_2 + A_3)] = D_2 + D_3 - D_1$, which can be derived directly from Eq. (13c), has been used.

## IV. Energy Release Rate

If the growth of a buckled delamination is governed by a Griffith-type criterion of a critical energy release rate, the prediction of whether delamination will grow requires an evaluation of energy release rate $G$. By applying the previous postbuckling solution, evaluating the total potential energy and differentiating the result with respect to the delamination length $2a$, we may obtain an explicit expression for $G$. The function $G(a)$ can then be used to study the initiation and stability of delamination growth.

The total potential energy $\Pi$ of the buckled delaminated sandwich consists of the contributions from strain energy $U$ due to bending and stretching of the faces, shearing of the core, and the potential energy $-W$ of the external forces, i.e.,

$$\Pi = U - W \tag{19a}$$

where

$$U = 2\left\{\frac{1}{2}\int_0^l (M_x\kappa_x + N_x\epsilon_{x0} + Q_x\gamma_{xz})\, dx\right\}$$
$$W = -2Pu_0(l) \tag{19b}$$

Note that the factor 2 is due to the symmetry condition considered in our problem. The expressions for $M_x$, $\kappa_x$, $N_x$, $\epsilon_{x0}$, $Q_x$, and $\gamma_{xz}$ can be found in the second equation of Eq. (14b); the first equations of Eqs. (4b), (4a), and (7); and Eq. (11). An alternate expression for $W$ is

$$W = -2\{P_1[u_0^{(1)}(l) - u_0^{(1)}(a)] + P_2u_0^{(2)}(a) + P_3u_0^{(3)}(a)\} \tag{19c}$$

where the superscript $(i)$ denotes the region number. By using the postbuckling solutions given in Eqs. (13-15), the final simplified result of the total potential energy can be expressed as

$$-\Pi = A_1P_1^2 l + 2\alpha a + 2\left(\beta - B_1\frac{P_1^2}{S}\right)\Gamma\lambda_1 \sin\lambda_1(l-a) + \frac{1}{2}\frac{P_1^2}{S}\Gamma^2\lambda_1 \sin 2\lambda_1(l-a) \tag{20a}$$

where

$$\alpha = ½(A_2P_2^2 + A_3P_3^2 - A_1P_1^2)$$
$$\beta = ½(B_2P_2 + B_3P_3 - B_1P_1) \tag{20b}$$

The energy release rate $G$ is then calculated as

$$G = -\frac{\partial\Pi}{2\partial a} = \alpha + \frac{\lambda_1^2}{2}\frac{P_1^2}{S}\Gamma^2 + \left\{\beta + [B_1 + \Gamma\cos\lambda_1(l-a)]\frac{P_1^2}{S}\right\} \times [\Gamma\lambda_1 \sin\lambda_1(l-a)]' \tag{21a}$$

where the prime (′) denotes differentiation with respect to the half delamination length $a$. From Eq. (15), we can also show that

$$[\Gamma\lambda_1 \sin\lambda_1(l-a)]' = \frac{1}{2\beta}[\Gamma\lambda_1 \sin\lambda_1(l-a)]^2\left[\frac{P_2}{\sin^2\lambda_2 a} + \frac{P_3}{\sin^2\lambda_3 a} - \frac{P_1}{\sin^2\lambda_1(l-a)}\right] \tag{21b}$$

## V. Special Cases

### A. Delaminated Composite (Without Core)

The problem of delamination buckling and growth in composite laminates has received a considerable amount of attention. Analytical investigation by a one-dimensional model has been done by several researchers.[2,3] The present results of composite sandwiches are also applicable to the cases of composite laminates (without core) by letting the terms containing $S$ vanish. The expressions for the postbuckling deflections and buckling load are exactly the same as Eqs. (13), (15), and (18) except that $\lambda_1^2$ and $\bar{\lambda}_1^2$ are now replaced by $\lambda_1^2 = P_1/D_1$ and $\bar{\lambda}_1^2 = P_{cr}/D_1$. The energy release rate shown in Eq. (21) is then simplified to

$$G = ½(A_2P_2^2 + A_3P_3^2 - A_1P_1^2) + ½[\Gamma\lambda_1 \sin\lambda_1(l-a)]^2\left[\frac{P_2}{\sin^2\lambda_2 a} + \frac{P_3}{\sin^2\lambda_3 a} - \frac{P_1}{\sin^2\lambda_1(l-a)}\right] \tag{22}$$

An algebraic expression for the energy release rate obtained by means of the path-independent $J$ integral has been presented by Sallam and Simitses.[9] That expression looks complicated and different from the one given in Eq. (22). However, by careful deduction one can prove that they are exactly the same.

### B. Perfect Sandwich (Without Delamination)

By letting the delamination length $2a$ approach zero, the characteristic equation for buckling load shown in Eq. (18) can be reduced to

$$\lim_{a\to 0}\frac{a}{\bar{\lambda}_1 \tan\bar{\lambda}_1(l-a)} = -\frac{D_1}{P_{cr}} \tag{23}$$

The solution to Eq. (23) exists only when $\bar{\lambda}_1 = \pi/l$. From the second equation of Eq. (18b) we have

$$P_{cr} = \frac{D_1(\pi/l)^2}{1 + (D_1/S)(\pi/l)^2} \tag{24}$$

which is a well-known solution for a perfect sandwich.[1] The buckling load for a perfect sandwich may be considered as an upper bound for the axial load capacity of a delaminated sandwich.

### C. Thin-Film Delamination

Because the faces are relatively thinner than the depth of the core, the delamination laid on the interface can usually be treated as thin-film delamination. Because the delamination is relatively slender in comparison with the whole plate, the buckling may be initiated by local buckling of the thin delamination. Since the thin layer of delamination has elastically supported ends, the buckling load $(1-k)P_{cr} = \bar{\lambda}_3^2 D_3$ is close to but less than that of a fixed end plate of length $2a$, i.e.,

$$(1-k)P_{cr} = \bar{\lambda}_3^2 D_3 \le \frac{4\pi^2 D_3}{(2a)^2}$$

or

$$\bar{\lambda}_3 \le \pi/a$$
$$P_{cr} \le \frac{1}{1-k}\frac{\pi^2 D_3}{a^2} \tag{25}$$

Consider the case when $\lambda_3 = \pi/a$; the postbuckling deflection shown in Eq. (13) can be reduced to

$$w_1 = w_2 = 0$$
$$w_3 = \Gamma^*[\cos(\pi x/a) + 1] \tag{26a}$$

where the amplitude $\Gamma^*$ is related to $\Gamma$ by

$$\Gamma^* = \Gamma \frac{\lambda_1 \sin \lambda_1(l-a)}{\lambda_3 \sin \lambda_3 a} \tag{26b}$$

The equilibrium and compatibility conditions obtained in Eq. (15) can also be simplified to, by using $\lambda_3 = \pi/a$ and Eq. (26b),

$$-\frac{P_3}{4D_3}\Gamma^{*2} + A_2P_1 - (A_2 + A_3)P_3 = 0$$
$$\Gamma^* = -[(B_2 - B_1)P_1 + (B_3 - B_2)P_3]/P_3 \tag{27a}$$

where

$$P_3 = D_3(\pi/a)^2 \tag{27b}$$

Equations (27a) and (27b) yield the unknown amplitude $\Gamma^*$ and the required postbuckling load $P_1$ to attain the special postbuckling deflection shown in Eq. (26). The energy release rate given in Eq. (21) for this special postbuckling deflection can also be reduced to

$$G = \alpha + \frac{\Gamma^{*2}P_3^2}{2\beta D_3}\left(\beta + B_1\frac{P_1^2}{S}\right) \tag{28}$$

If we consider a homogeneous orthotropic plate of thickness $t$ containing a parallel plane crack at a depth $h$ from the top surface of the plate, the previous results in Eqs. (25–27) can be shown to be exactly the same as those presented by Yin et al.[3] Similar to the cases discussed in Ref. 3, a lower bound of the ultimate load capacity is given by the combined axial load capacity of two detached plates.

## VI. Results and Discussions

In the following examples, graphite/epoxy lamina is selected to construct the faces of sandwich whereas aluminum honeycomb is used to be the core of the sandwich. The half-length of sandwich beam $l$ is 50 mm. The material properties of graphite/epoxy are[7] $E_{11} = 181$ GPa, $E_{22} = 10.3$ GPa, $G_{12} = 7.17$ GPa, and $\nu_{12} = 0.28$ where $E$, $G$, and $\nu$ are the Young's modulus, shear modulus, and Poisson's ratio, respectively. The subscript 1 denotes the fiber direction and the subscript 2 denotes the transverse direction. The thickness of each lamina $t_{ply}$ is 0.125 mm. The reference effective transverse shear moduli of aluminum honeycomb selected are $G_{xz_0} = 0.146$ GPa, $G_{yz_0} = 0.0904$ GPa, and the thickness of honeycomb core $c_0$ is 10 mm.[7]

### A. Core Effect on Buckling Load

The effect of core on the buckling load is usually discussed by considering the thickness and effective transverse shear modulus separately. However, in our formulation only one parameter $S(=cG_{xz})$ needs to be considered. An example of $[0_2/90_2/9_2/\text{honeycomb}]_s$ with $S_0 = 1.46$ MN/m is used to study this effect. The results are presented by the relative buckling load $P_{cr}/P_{cr}^c$ where $P_{cr}^c$ denotes the buckling load of delaminated composite (without core). Figure 2 shows that the

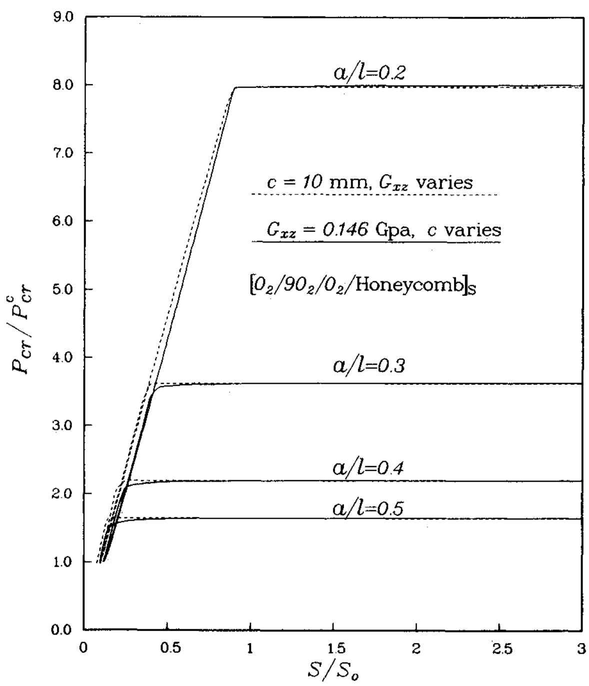


Fig. 2 Effect of core on buckling load.

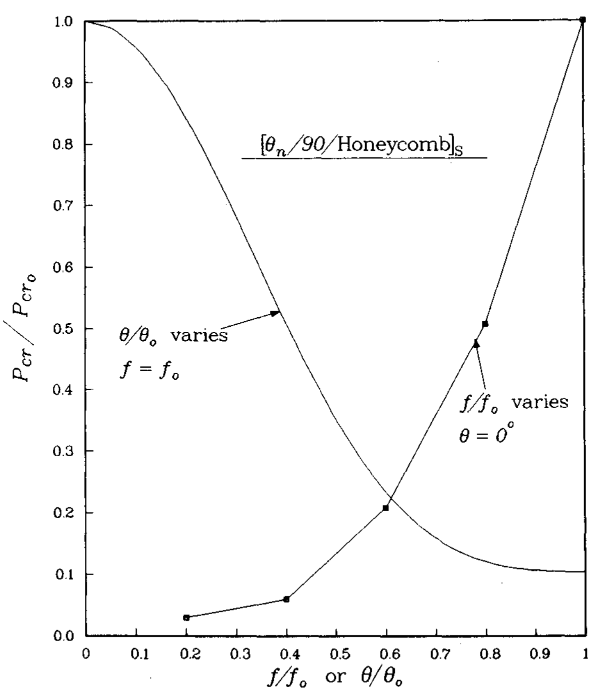


Fig. 3 Effect of face on buckling load.

**Table 1 Effect of stacking sequence on buckling load of delaminated sandwich [composite laminate/honeycomb]$_s$**

| | $P_{cr}/P_{cr_0}$ | | | | |
|---|---|---|---|---|---|
| $a/l$ | $ss1$[a] | $ss2$[a] | $ss3$[a] | $ss4$[a] | $ss5$[a] |
| 0.2 | 0.2598 | 0.1772 | 0.1484 | 0.1476 | 0.1131 |
| 0.3 | 0.1155 | 0.0788 | 0.0660 | 0.0656 | 0.0503 |
| 0.4 | 0.0650 | 0.0431 | 0.0371 | 0.0369 | 0.0283 |
| 0.5 | 0.0416 | 0.0284 | 0.0238 | 0.0236 | 0.0181 |

[a]$ss1 \equiv [0°/-45°/45°/0°]$, $ss2 \equiv [-45°/0°/45°/0°]$, $ss3 \equiv [45°/-45°/0°/0°]$, $ss4 \equiv [0°/0°/-45°/45°]$, $ss5 \equiv [-45°/0°/0°/45°]$.

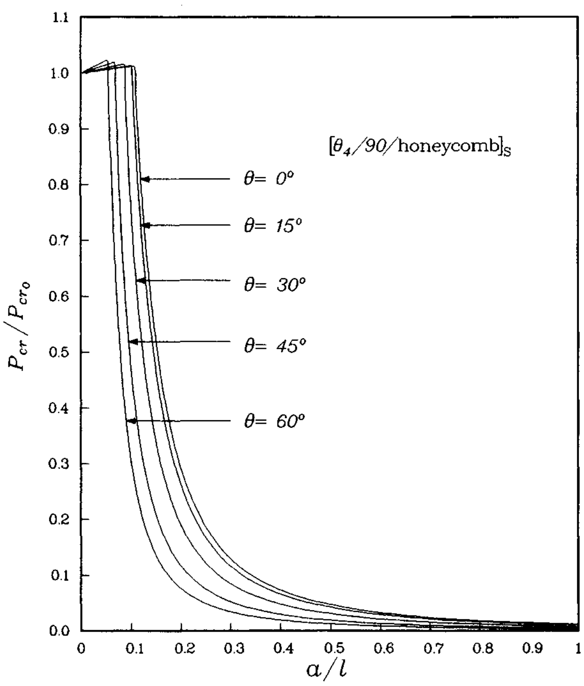


**Fig. 4 Effect of delamination length on buckling load.**

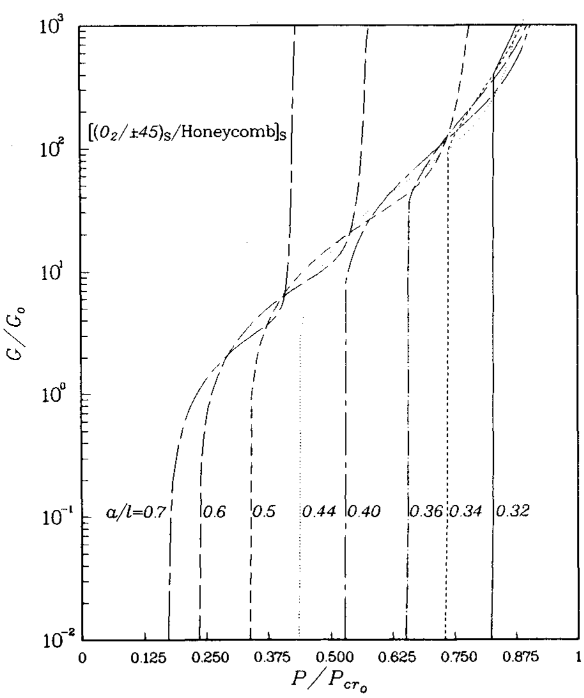


**Fig. 5 Energy release rate vs applied compressive force with delamination length given.**

buckling load will increase when the transverse shear stiffness $S$ increases until it reaches a certain value and becomes a constant, which can be explained as thin-film delamination buckling, and is verified by Eq. (25). The solid line is under the condition $G_{xz} = G_{xz_0}$ with $c$ varied whereas the dot line is under the condition $c = c_0$ with $G_{xz}$ varied. It shows that no matter how we change the transverse shear stiffness by varying core thickness $c$ or transverse shear modulus $G_{xz}$, the buckling loads for both cases are almost the same. The difference comes from $D_1$, $D_2$, and $D_3$ in Eq. (18), which are related to the core thickness but are independent of the transverse shear modulus. Figure 2 also gives us a minimum value of $S$ that makes the buckling load of delaminated sandwich larger than that of delaminated composite. Below this value, it may not be a useful construction for sandwich since the core thickness may be smaller than face thickness and the usual assumption for sandwich listed in Sec. II may not be valid. One should note that the special case discussed in Sec. V.A is reduced by letting the terms containing $S$ vanish, not by letting $S$ be zero.

### B. Face Effect on Buckling Load

To discuss the effect of face, we consider the thickness, fiber direction, and stacking sequence of the laminated composite. Before the calculation, we may expect that buckling load will increase when the face thickness increases, or when the fiber is oriented to the direction of load (i.e., $\theta$ varies from 90 to 0 deg). Figure 3 verifies this expectation. A series of composite sandwiches $[\theta_n/90/\text{honeycomb}]_s$ are used in this case. The reference parameters are $f_0 = 0.625$ mm (i.e., $n = 4$), $\theta_0 = 90$ deg, and $a = 0.2l$; and $P_{cr_0}$ is the buckling load corresponding to $f = f_0$ with $\theta = 0$ deg. The thickness effect is considered by varying the number of $\theta = 0$ deg lamina, i.e., changing $n$ from 0 to 4.

To study the effect of stacking sequence, we consider five different sequences shown in Table 1 where $P_{cr_0}$ represents the buckling load of perfect composite sandwich. For different delamination length, Table 1 shows that the buckling load of delaminated sandwich with stacking sequence $ss1$ is always the highest one, whereas that of $ss5$ is always the lowest one. Since $ss1$ is not the one with highest bending stiffness $D_{11}$ and $ss5$ is not the one with lowest bending stiffness, the effect of coupling stiffness $B_{11}$ is revealed in this example. The order of the buckling load shown in this table is $ss1 > ss2 > ss3 > ss4 > ss5$, which is consistent with the order of $D_3$ (or $D_2$), not $D_1$. This also means that the example considered can be approximated by the thin-film delamination as shown in Eq. (25).

### C. Effect of Delamination Length on Buckling Load

The sandwich construction $[\theta_4/90/\text{honeycomb}]_s$ is used as an example to study the delamination length effect on buckling load. Figure 4 shows the relation between the relative buckling load $P_{cr}/P_{cr_0}$ and delamination length $a/l$ where $P_{cr_0}$ denotes the buckling load of perfect sandwich. As expected, the upper bound solution is approximated by the buckling load of perfect sandwich shown in Eq. (24), whereas the combined axial load of two completely detached beam represents the lower bound. Figure 4 also shows that the buckling loads for some short delaminations are a little higher than $P_{cr_0}$, which seems unreasonable at a first glance but the same phenomenon has been found in experiments for delaminated composites.[17,18]

### D. Delamination Growth

The postbuckling analysis assumes that the fracture toughness of the material is sufficiently large to resist the growth of delamination. When delamination growth does occur, the geometry of the problem is irreversibly changed. However, owing to the intrinsic complexities involved, such as anisotropy, inhomogeneity, and noncontinuity, a satisfactory, physically meaningful, and universal delamination failure criterion has not yet been found. Here, we assume that delamination growth is governed by a critical total energy release rate $G_0$. For a

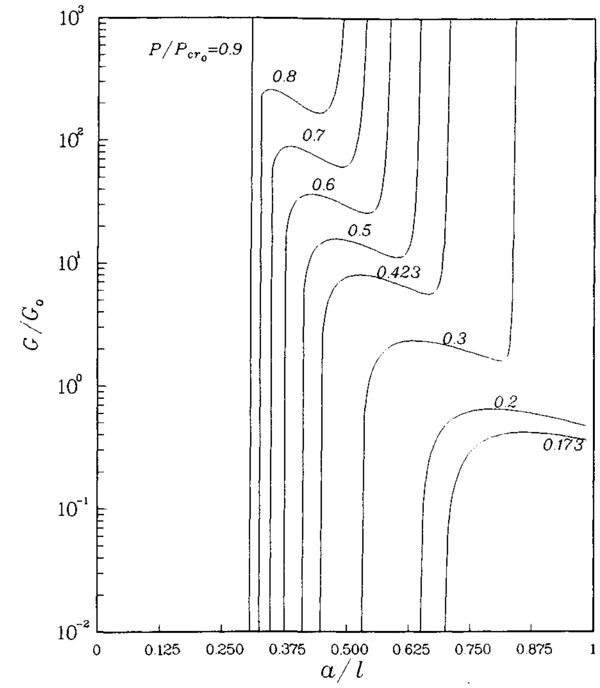


Fig. 6 Energy release rate vs delamination length with applied compressive force given.

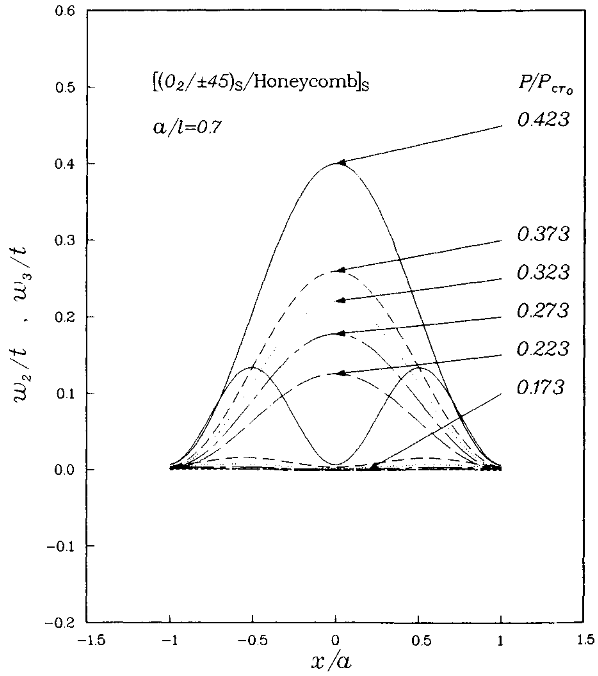


Fig. 7 Postbuckling deformation of the delaminated composite sandwich with $a/l = 0.7$.

specified delamination length, the buckling load $P_{cr}$ can be obtained from Eq. (18), and the associated energy release rate immediately after buckling is calculated in Eq. (21). If the energy release rate is less than the critical value $G_0$, the delamination will not grow and the ultimate axial load capacity $P_{ult}$ of the delaminated sandwich will be larger than $P_{cr}$, otherwise, $P_{ult} = P_{cr}$. However, the actual $G_0$ is still unknown. For the purpose of explaining the delamination growth, we take $G_0$ as the critical total energy release rate of the composite laminate used in the sandwich faces. In our examples, the sandwich is of the type $[(0_2/\pm 45)_s/\text{honeycomb}]_s$; and the faces of sandwich considered are $[0_2/\pm 45]_s$ made of T300/5208 graphite/epoxy whose critical value $G_0 = 460$ J/m$^2$ (Ref. 19) and $E_{11} = 137.9$ GPa, $E_{22} = 14.5$ GPa, $G_{12} = 5.9$ GPa, $\nu_{12} = 0.21$, and $t_{ply} = 0.14$ mm.

Figure 5 shows that, for a given delamination, the energy release rate increases when the applied axial load increases in which $P_{cr_0}$ is the buckling load of perfect sandwich. Generally, each curve for a fixed $a/l$ has two steep slopes. One is at the beginning, which corresponds to the buckling load; the other is near the end, which may be responsible for the delamination growth. For example, if $a/l = 0.7$, there is a threshold value of $P/P_{cr_0} = 0.173$ below which the energy release rate is identical to zero. This means that the threshold value of $P$ is the buckling load $P_{cr}$. The energy release rate associated with the load immediately after buckling is approximated to $0.2G_0$, which is less than the critical value $G_0$. Hence the delamination will not grow until the load is increased to $0.232P_{cr_0}$ whose energy release rate is $G_0$, and $P_{ult} = 0.232P_{cr_0} > P_{cr} = 0.173P_{cr_0}$. The same approach can be taken to find $P_{cr}$ and $P_{ult}$ if the given diagram is under fixed load condition such as Fig. 6. The corresponding postbuckling deformation for $a/l = 0.7$ is shown in Fig. 7. Under critical buckling state, i.e., $P = P_{cr} = 0.173P_{cr_0}$, the deflections of both upper and lower segment are still flat. However, a small disturbance will make the sandwich buckle. When constant load increment $0.05P_{cr_0}$ applies, the deflection of region 3 is convex upward whereas that of region 2 deforms a little. There are two stages that have great jump in the deformation. One occurs near the critical buckling stage and the other happens when $P = 0.423P_{cr_0}$ from which the energy release rate increases rapidly. This phenomenon reveals that the actual $G_0$ may be higher than the chosen one.

## VII. Conclusions

A mathematical model for the delaminated composite sandwich has been developed in this paper. By applying the one-dimensional formulations, the transverse deflection and bending moment of the postbuckling solution are obtained. The explicit closed-form solutions of the critical buckling load and energy release rate are also derived based on this postbuckling result. Some special cases such as delaminated composites, perfect sandwiches, and thin-film delaminations are used to verify our solutions. The effects of core, face, and delamination length on the buckling load, the delamination growth, and the ultimate axial load capacity of the delaminated composite sandwiches are also discussed in this paper. The results show the following. 1) The buckling load will increase when the transverse shear stiffness increases until it reaches a certain value and becomes a constant. 2) The buckling load will increase when the face thickness increases or the fiber is oriented to the direction of load. 3) The upper bound solution of the delaminated composite sandwiches is approximated by the buckling load of perfect sandwich whereas the combined axial load of two completely detached beams represents the lower bound. 4) If the energy release rate associated with the load immediately after buckling is less than the critical energy release rate, the delamination will not grow and the ultimate axial load capacity $P_{ult}$ of the delaminated composite sandwiches will be larger than the buckling load $P_{cr}$; otherwise $P_{ult} = P_{cr}$.

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